Capita selecta in engineering mechanics

# Work, energy methods & influence lines

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## Colophon

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# Work, energy methods & influence lines

## Preface

The content of this textbook is a compilation of my lecture notes used at the Delft University of Technology, faculty of Civil Engineering. The topics in this volume are *Work and energy principles* and *Influence lines*. A vast amount of extensive books have been written on these subjects and as a teacher of an undergraduate course in Structural Mechanics, I have to focus on the essentials. Over the years I therefore wrote a number of notes in which the topics are covered in such a manner that students can study the material themselves and prepare the assignments which are discussed in class. Theory and application are therefore directly combined and numerous examples are added in order to clarify all steps involved. This approach is highly appreciated by students and based on the feedback of the students the notes were further improved.

So far most methods used in the under graduate courses on Structural Mechanics are based on direct methods to find the force distribution in structures and or to determine the deformations and displacements. Well known classical methods are:

- Equilibrium method to find the force distribution in statically determinate structures,
- Moment-area theorems to compute the deflections,
- Euler Beam theory by solving the fourth order differential equation to find both the force distribution and the displacements in statically (in)determinate structures,
- Practical application of engineering equations (*forget –me-not's*) which can be found by either of the two previous indicated methods to find displacements and or rotations in beam type structures
- Force method to find the force distribution in statically indeterminate structures.
- Displacement method to find the displacements and thus the force distribution in statically indeterminate structures.

Next to these methods a host of alternative methods exists based on work and energy principles. These lecture notes will introduce these. To understand these methods and to see the difference in application it is essential to understand the previous mentioned classical methods. An overview of these can be found in the standard study books on mechanics such as our own series of books by Hartsuijker and Welleman<sup>[3-5]</sup>. Although all methods described here are from the past and new computer tools based on the finite element method will be used in engineering practise, these old methods are still important. An advantage of today's symbolic algebra tools like MAPLE is that solving problems with these old styled methods have become much more attractive and give more qualitative insight in the solutions. All examples in these notes can therefore easily be solved with the use of MAPLE and the reader is urged to do so.

#### **Overview of topics**

The topics are covered in five chapters. In the *chapter 1* some introducing remarks are made and the concept of work and virtual work is explained. The principle of virtual work has been introduced in the first year courses on mechanics and is an alternative method to find the force distribution in statically determinate structures. In *chapter 2* the deformation or strain energy is introduced which is used in *chapter 3* in finding the Castigilano theorem. *Chapter 4* describes a more generalised approach based upon the principle of minimum potential energy which will be used in approximation methods. *Chapter 5* introduces the concept of influence lines for both static determinate and indeterminate structures. In order to fully understand the concept of an influence line, knowledge of work and virtual work is required.

#### Acknowledgement

I made grateful use of published work of other authors<sup>[1-2; 7-10]</sup> and (former) colleagues. Most of this material was published in Dutch as 'collegedictaat' (lecture notes). This new collection of notes in English provides both the Dutch and international students with a set of notes which will introduce them into the topics of work and energy methods with applications on influence lines.

The aim of this book is to present a condensed and comprehensive introduction into work and energy methods, and influence lines. This book is not a complete reference but primarily meant to introduce the topics to undergraduate students in (civil) engineering. With this introduction the reader is able to further study the established literature on work and energy methods at a (post) graduate level and the reader is kindly invited to study these books.

A special recognition goes to Coen Hartsuijker – in this book I frequently refer to his books and used with permission part of his Dutch notes – and to Cor van Eldik, the publisher at Bouwen met Staal, who helped me out with producing this book.

Hans Welleman August 2016

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## Work and energy

If an elastic body is loaded by external forces these forces produce work (figure 1.1). The body is deformed and thus the point of application of the force will move. What is the result of this work? Inside the body deformations will occur and the external applied work will be stored inside the material as deformation or strain energy. If the body is unloaded, this energy will be released and the body returns to its original shape. This chapter deals with the relation between work and energy.



1.1 Elastic body loaded by an external force.

#### 1.1 Work generated by forces

Work is defined as the product of a force times the *associated displacement*. To elaborate on this the associated displacement  $u_F$  is shown in figure 1.2 as the displacement component along or associated to the applied force. In vector notation work is the *dot product* of the two vectors  $\vec{F}$  and  $\vec{u}$ .



1.2 Work done by a force.

Work can also be negative. In that case the force and displacement are opposed to each other. From the figure it also becomes clear that a displacement perpendicular to the force will not contribute to the amount of work.

#### **1.2** Work generated by a couple (moment)

A couple can also produce work since a couple is a system of forces. In figure 1.3 both a force F and a couple T are applied at the indicated point of application. Assume only a rotation at the point of application.





Both the force and the couple can be replaced by a single force F at a distance a from the original point of application. The work produced by the force due to a small rotation can be expressed in terms of the original couple T:

$$A = F \times u = \frac{T}{a} \times u = T \times \frac{u}{a} = T \times \varphi$$

Work done by a couple is therefore equal to the product of the moment of the couple times the angle of rotation. Although not proven here, this also holds for large rotations.

The force will rotate due to rotations. In figure 1.3 this is simplified. If however the rotation of the force is taken into account the proof for large rotations can be found, this is left to the reader.

#### **1.3** Coordinate system and units

In these notes primarily planar structures will be used. The coordinate system used is the *x-z* plane. This plane is shown in figure 1.4. A positive rotation around the *y*axis is denoted with  $\varphi$ . If no coordinate system is specified it is either not required or an *x-z coordinate system* is assumed. *Forces* are expressed in [kN], *lengths* in [m].



1.4 Coordinate system.

#### 1.4 Work and deformation, Clapeyron's law

If work is applied to an *elastic* body the body will deform. During deformation the increase in work is stored as deformation or strain energy. Strain energy and work are both expressed in joule and are equivalent quantities  $[J = joule = newton \times meter]$ .



In this chapter the expression for the deformation or strain energy will be determined for each basic load carrying principle:

- axial loading (tension or compression)
- shear
- bending
- torsion

For each load case one generalised stress (sectional force) is considered. (e.g. normal force N, shear force V, bending moment M and a torsional moment  $M_t$ ). Apart from these basic (generalised) load cases we can also look to specific stress situations like:

- normal stresses
- shear stresses

#### 2.1 Axial loading

A typical axial load case is shown in figure 2.1 An elastic material behaviour is assumed.



2.1 Basic axial loading.

The constitutive relation which relates the internal (generalised) stress (normal force *N*) to the deformation (strain)  $\varepsilon$  (epsilon) is presented with the *N*- $\varepsilon$  diagram of figure 2.1.

The small element with length dx is strained by the axial force N. The strain  $\varepsilon$  is shown in the force-strain diagram. An increase d $\varepsilon$  of this strain will elongate the element by:

 $dl = d\varepsilon dx$ 

The axial force will hardly be affected by this increase in strain and is assumed to be constant. The change of work generated by this normal force can be expressed as:

 $dA = Ndl = Nd\varepsilon dx$ 

Per unit of length this results in:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = N\mathrm{d}\varepsilon$$

This work is stored as strain energy according to Clapeyron. The increase of strain energy per unit of length thus becomes:

$$\mathrm{d}E_v = \frac{\mathrm{d}A}{\mathrm{d}x} = N\mathrm{d}\varepsilon$$

For a specific normal force which belongs to a certain strain level the total amount of strain energy per unit of length can be computed with:

$$E_v^* = \int_o^{\varepsilon} dE_v = \int_o^{\varepsilon} Nd\varepsilon \quad (\text{ per unit of length is marked by }^*)$$

In case of a **linear elastic** material behaviour the constitutive model by Hooke can be used:

$$N = EA\varepsilon$$

Substituting this relation in the expression for the strain energy results in:

$$E_{\nu}^{*} = \int_{0}^{\varepsilon} EA\varepsilon d\varepsilon = EA \int_{0}^{\varepsilon} \varepsilon d\varepsilon = EA \Big| \frac{1}{2} \varepsilon^{2} \Big|_{0}^{\varepsilon} = \frac{1}{2} EA\varepsilon^{2}$$

This result can also be expressed in terms of the generalised stress and is then denoted as *complementary strain energy*  $E_c$ :

$$E_v^* = E_c^* = \frac{N^2}{2EA}$$



In section 2.7 the deflection at the point of application of a force was computed based upon the strain energy stored in the entire beam. In this chapter a more general applicable method will be derived which was introduced by Cotterill and later Castigliano around 1870.

#### 3.1 Castigliano's second theorem

In figure 3.1 a beam is shown loaded by a series of concentrated loads.



All loads gradually increase up to the end value  $F_i$  generating a total external amount of work:

$$A_{ext} = \frac{1}{2}F_a u_a + \frac{1}{2}F_b u_b + \frac{1}{2}F_c u_c + \dots + \frac{1}{2}F_x u_x$$
(3.1)

In this expression the displacement at the point of application of each load is used. These displacements can however be expressed in terms of the total applied load as was shown in section 1.6. For the loaded beam shown in figure 3.1, this results in:

$$u_{a} = c_{aa}F_{a} + c_{ab}F_{b} + c_{ac}F_{c} + \dots + c_{ax}F_{x}$$

$$u_{b} = c_{ba}F_{a} + c_{bb}F_{b} + c_{bc}F_{c} + \dots + c_{bx}F_{x}$$

$$u_{c} = c_{ca}F_{a} + c_{cb}F_{b} + c_{cc}F_{c} + \dots + c_{cx}F_{x}$$

$$u_{x} = c_{xa}F_{a} + c_{xb}F_{b} + c_{xc}F_{c} + \dots + c_{xx}F_{x}$$
(3.2)

The coefficients  $c_{ij}$  are the influence factors or components of the *flexibility matrix* as introduced by Maxwell. By differentiating the external work with respect to a specific load  $F_x$  at x we obtain:

$$\frac{\partial A_{ext}}{\partial F_x} = \frac{1}{2} F_a \frac{\partial u_a}{\partial F_x} + \frac{1}{2} F_b \frac{\partial u_b}{\partial F_x} + \frac{1}{2} F_c \frac{\partial u_c}{\partial F_x} + \dots + \frac{1}{2} u_x + \frac{1}{2} F_x \frac{\partial u_x}{\partial F_x}$$
(3.3)

**Note.** To obtain the last term the rules of differentiating have to be applied in a proper way, check this.

The partial derivatives can be found using expression (3.2):

$$\frac{\partial u_{a}}{\partial F_{x}} = \frac{\partial (c_{aa}F_{a} + c_{ab}F_{b} + c_{ac}F_{c} + \dots + c_{ax}F_{x})}{\partial F_{x}} = c_{ax}$$

$$\frac{\partial u_{b}}{\partial F_{x}} = \frac{\partial (c_{ba}F_{a} + c_{bb}F_{b} + c_{bc}F_{c} + \dots + c_{bx}F_{x})}{\partial F_{x}} = c_{bx}$$

$$\frac{\partial u_{c}}{\partial F_{x}} = \frac{\partial (c_{ca}F_{a} + c_{cb}F_{b} + c_{cc}F_{c} + \dots + c_{cx}F_{x})}{\partial F_{x}} = c_{cx}$$

$$\frac{\partial u_{x}}{\partial F_{x}} = \frac{\partial (c_{xa}F_{a} + c_{xb}F_{b} + c_{xc}F_{c} + \dots + c_{xx}F_{x})}{\partial F_{x}} = c_{xx}$$
(3.4)

Substituting this result in expression (3.3) results in:

$$\frac{\partial A_{ext}}{\partial F_x} = \frac{1}{2} F_a c_{ax} + \frac{1}{2} F_b c_{bx} + \frac{1}{2} F_c c_{cx} + \dots + \frac{1}{2} F_x c_{xx} + \frac{1}{2} u_x$$
(3.5)

According to Maxwell's reciprocal theorem, see also section 1.6:

$$c_{ax} = c_{xa}$$

$$c_{bx} = c_{xb}$$

$$c_{cx} = c_{xc}$$
(3.6)

Using this result in expression (3.5) we obtain:

$$\frac{\partial A_{ext}}{\partial F_x} = \underbrace{\frac{1}{2}c_{xa}F_a + \frac{1}{2}c_{xb}F_b + \frac{1}{2}c_{xc}F_c + \dots + \frac{1}{2}c_{xx}F_x}_{\frac{1}{2}u_x} + \frac{1}{2}u_x$$
(3.7)

Expression (3.7) can be simplified as:

$$\frac{\partial A_{ext}}{\partial F_x} = u_x \tag{3.8}$$

This result shows that differentiating the total external work with respect to a specific load will result in the associated displacement at the point of application of this specific load. This principle is known as the **second theorem of Castigliano** although **Cotterill** found this result a few years earlier <sup>[11]</sup>.

# Energy functions and approximations

Castigliano's theorems have been introduced in the previous chapter based on linear elasticity. In this chapter a more comprehensive approach will be introduced with help of energy functions in which also the validity of the previously found theorems is being discussed. An application of energy functions is the approximation method based upon minimum potential energy. With a few examples this method will be demonstrated in this chapter.

#### 4.1 Energy function

Virtual work has been introduced in chapter 1 as an alternative description of the equations of equilibrium since zero virtual work contains the equations of equilibrium:

$$\delta A = 0$$

In words this principle states that due to a kinematically admissible *virtual displacement* field the sum of all *virtual work* generated by all forces must be equal to zero.

If a virtual displacement is considered as a small perturbation of an equilibrium situation the principle of virtual work states that the variation of the virtual work due to this perturbation is equal to zero. In fact this requires a stationary situation of the work function. This problem can be visualized with the following analogy. In figure 4.1 a ball is placed on a friction less path. This path can be described as a function. The function value is stationary for three indicated positions of the ball.



4.1 Stationary position.

The path of the ball can be regarded as an *energy or work function*. In the previous chapters we already have seen that these two are exchangeable quantities. We therefore introduce an energy potential function *V*. Figure 4.1 shows several

positions for which the potential function is stationary. The top left position is however not a stable position. For perturbations to the left the ball will not return to its original equilibrium position. The second position is better since this *local extreme* results in a stable equilibrium. However the position at the far right results in a *stable position* at the *lowest value* of the potential function. This latter aspect of the global minimum will be addressed later in this chapter.

In order to ensure a stable equilibrium situation the energy function must be stationary at a minimum. This results in:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 0$$
 and  $\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} > 0$ 

The components of *V* are yet to be determined. From the conservation of energy it is known that work generated by forces will be transferred into *movements*, *deformations* and or *heat*. These are labelled by *kinematic* energy, *strain* energy and *dissipation* energy. For elastic materials all stored energy is released during unloading, therefore no dissipation occurs.

According to the principle of conservation of energy the total amount of energy in a closed environment remains constant:

#### $E_{kin} + E_{pot} = constant$

If statics is considered *kinematic energy* is of no concern since accelerations are not taken into account. The work generated by forces can only be stored as *strain energy*. This reduces the conservation of energy to:

#### $E_{pot} = constant$

The energy function we are looking for is in this particular case (statics and elasticity) a *potential energy function*. As a result of figure 4.1 a *variation of the potential energy function* should be *equal to zero* for all (kinematically) possible perturbations of state variables. Physically this can be regarded as a stationary energy level for small perturbations of state variables. Possible state variables will be demonstrated later in a few examples.

The possible *exchange* of energy can only be between the contributions to the potential energy. On one hand there is *energy from work generated by forces* which is stored as *strain energy* in an *elastic material*. Both components can be regarded as potential energy. We therefore have to make a distinction between *potential* 



# **Influence lines**

In this chapter the concept of influence lines will be introduced. First the concept of influence lines will be explained followed by the description of tools to construct influence lines of statically determinate and indeterminate systems for both force and displacement quantities. The theory will be applied to numerous examples.

#### 5.1 Problem sketch and assumptions

In courses on Statics the focus is primarily on finding the force distribution in a structure as a result of a static load at a fixed location. This force distribution can be visualised using normal-, shear- and moment-diagrams. If the position of the load is variable as shown in figure 5.1 there is a problem in visualising the force distribution.



5.1 Unit load that moves over the structure.

To find the magnitude of a force or displacement at a fixed position depending on the location of the external load the concept of *influence line* is introduced. An influence line gives the value of a quantity in a specific point as a function of the location of a moving unit load.

The *unit load* is a concentrated load with magnitude 1.0. Influence lines will be used to find the most *unfavourable position* of the load for a certain quantity in statically determinate and statically indeterminate systems. Quantities of interest could be:

- a support reaction
- a rotation at a certain point
- an internal moment in a certain cross-section
- a deflection at a specific point
- a shear force in a specific cross-section

In this chapter we will discuss how to determine influence lines for statically determinate and indeterminate systems. In the last part the *most unfavourable position* of the load for a certain quantity will be determined. For some applications only the shape of the influence line is important. This is referred to as the *qualitative aspect* of the influence line. In other cases also the exact magnitude of the influence factor is relevant. This latter aspect is referred to as the *quantitative aspect* of the influence line.

#### 5.1.1 Introduction of influence lines for force quantities

The concept of influence lines is introduced with the following beam structure. The beam in figure 5.2 is loaded with a concentrated load of 1.0 kN that moves from the left support to the right support. This moving load is denoted with the horizontal dotted arrow.



5.2 Moving unit load on a simply supported beam.

The *dynamic* effects caused by the moving load are neglected completely. The problem is considered to be a *static problem*. Using this example we will introduce the influence lines for force quantities:

- influence lines for support reactions  $A_V$  and  $B_V$  (positive reactions assumed upwards)
- influence line for (internal) shear force  $V_{\rm C}$
- influence line for (internal) moment  $M_{\rm C}$

#### Influence line for support reactions at A and B

The question is how the support reactions  $A_V$  and  $B_V$  are related to the position of the moving load. *Equilibrium* yields to:

$$A_{\rm v} = \frac{(l-x) \times 1.0}{l} \qquad B_{\rm v} = \frac{x \times 1.0}{l}$$

The distribution of the support reaction at A and B depending on the position of the load x is shown in figure 5.3.

These graphs are called the influence lines for  $A_V$  and  $B_V$ . To determine the support reaction  $A_V$  for a load F placed somewhere on the structure we have to multiply the influence factor found from the influence line at that location with the magnitude of the concentrated load. A load of 50 kN at 2.5 meters from the left support results in a support reaction  $A_V$  at A of:

$$A_{\rm V} = 0.75 \times 50 = 37.5 \,\rm kN$$

This support reaction acts upwards. It is important to take care of the *signs* and to pay attention to the *definition* of the chosen *coordinate system*. Here we have chosen to plot the positive values (upwards acting) of the support reaction *downwards in the graph*.